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The separation of Newtonian shock layers

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The separation points which occur in the Newtonian theory of hypersonic flow are treated by locally modifying the shock-layer equations. This approach leads to direct verification of the free-layer theory for separation on certain bodies with discontinuous curvature. Separation points on bodies with continuous curvature have already been treated by matching the upstream shock-layer solution to the downstream free-layer solution. The agreement that is found between these results and those of the present approach provides further confirmation of the free-layer theory.

1. Introduction

In the Newtonian theory of hypersonic flow, Hayes & Probstein (1959) and Lighthill (1957) have suggested that the solution downstream of a separation point takes the form of a 'free layer'. This is a high-density layer close to the shock wave, but separated from the body by a very-low-density region. The first approximation to the free-layer's shape is given by the condition that the pressure along its base should vanish.

The plausibility of the free-layer theory has been demonstrated by Freeman (1960) for the case of the separation point on a sphere. Comparison of the shock-layer solution upstream of the separation point with the free-layer solution downstream enables the orders of magnitude of the flow variables to be found in the separation region itself as γ, the ratio of specific heats of the gas, tends to unity. By scaling with respect to these orders of magnitude and then letting

$$\epsilon = (\gamma-1)/(\gamma+1) \to 0,$$

an ordinary differential equation for the shock shape can be deduced which is uniformly valid throughout the separation region. In particular, the distance from the body to the shock is found to be of order $\epsilon^{\frac{9}{11}}$. The same technique has also been applied to other body shapes by Bausch (1962).

Freeman also suggested that the singularity in Newtonian shock-layer theory at a separation point results from neglecting the curvature of the streamlines relative to the body. We shall thus consider separation points from a different point of view, namely by modifying the shock-layer equations and not using the notion of a free layer at all. The agreement we shall find between this approach and the results of Freeman and Bausch may then be regarded as indirect verification of the free-layer theory.

Newtonian separation points occur on convex bodies which either have sufficiently large continuous curvature or else a sufficiently large curvature discontinuity, and we shall term these natural and artificial separation points, respectively. We shall first use the modified shock-layer equations to obtain direct agreement with the free-layer theory for a simple case of artificial separation, and then apply them to natural separation points and in particular the case of the sphere. We shall, however, exclude body shapes which are so blunt that the shock-layer solution upstream of the separation point cannot be found by a series expansion in powers of ϵ (Freeman 1956).

2. The shock-layer equations near separation points

Suppose a uniform inviscid perfect-gas stream with velocity U_∞ and density ρ_∞ flows at infinite Mach number past a two-dimensional or axisymmetric convex obstacle. In the region between the shock wave and the body let p, ρ, u

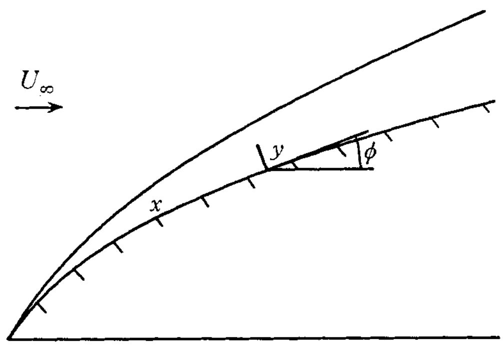

FIGURE 1. The curvilinear co-ordinate system.

and v be the pressure, density and tangential and normal velocity components, non-dimensionalized with respect to $\rho_\infty U_\infty^2$, ρ_∞ and U_∞ respectively. If x and y are the curvilinear co-ordinates shown in figure 1, and ψ is the stream function, the equations of motion with x, ψ as independent variables are

$$u\frac{\partial u}{\partial x}+\kappa uv+\frac{1}{\rho}\left(\frac{\partial p}{\partial x}-hr^j\rho v\frac{\partial p}{\partial \psi}\right) = 0, \tag{2.1 a}$$

$$\frac{\partial v}{\partial x}-\kappa u+hr^j\frac{\partial p}{\partial \psi} = 0, \quad \frac{\partial}{\partial x}(p/p^\gamma) = 0, \tag{2.1 b, c}$$

$$\frac{\partial y}{\partial \psi} = \frac{1}{r^j\rho u}, \quad \frac{\partial y}{\partial x} = \frac{hv}{u}, \tag{2.1 d, e}$$

where κ is the body curvature, $h = 1+\kappa y$, r is distance from the axis and $j = 0$ or 1 in two-dimensional and axisymmetric flow respectively. The boundary conditions are

$$y = 0 \quad \text{on} \quad \psi = 0 \tag{2.2 a}$$

and

$$u = \cos(\phi+\delta)\cos\delta+\epsilon\sin(\phi+\delta)\sin\delta, \tag{2.2 b}$$

$$v = \epsilon\sin(\phi+\delta)\cos\delta-\cos(\phi+\delta)\sin\delta, \tag{2.2 c}$$

$$p = (1-\epsilon)\sin^2(\phi+\delta), \quad \rho = 1/\epsilon, \tag{2.2 d, e}$$

on the shock $y = y_s(x)$, where $\tan\delta = y_s'(x)$, $\phi(x)$ is the body inclination and $\epsilon = (\gamma-1)/(\gamma+1)$. The first-order Newtonian solution is found by putting $u = u_0$, $v = \epsilon v_0$, $p = p_0$, $\rho = \epsilon^{-1}\rho_0$, $y = \epsilon y_0$ and letting $\epsilon \to 0$. Hence

$$u_0 = \cos\phi(\zeta), \tag{2.3a}$$

$$p_0 = \sin^2\phi(x) + \frac{\kappa(x)}{R^j(x)}\int_{\Psi(x)}^{\psi}\cos\phi(\zeta)\,d\psi, \tag{2.3b}$$

$$\rho_0 = p_0/\sin^2\phi(\zeta), \tag{2.3c}$$

$$y_0 = \frac{1}{R^j(x)}\int_0^{\psi}\frac{d\psi}{\rho_0 u_0}, \tag{2.3d}$$

$$v_0 = u_0\,\partial y_0/\partial x, \tag{2.3e}$$

where ζ is the x co-ordinate of the intersection of the streamline through (x, ψ) with the shock wave as $\epsilon \to 0$, and $\psi = 1/(j+1)\,R^{j+1}(\zeta) = \Psi(\zeta)$, where $R(x)$ is the body thickness. We shall exclude body shapes which are so blunt that (2.3*d*) does not converge, at least in a region sufficiently near the nose.

If $p_0(x_0, 0) = 0$ on a body with continuous curvature, then we shall call $x = x_0$ a natural separation point. Any curvature discontinuity at $x = x_0$ produces a discontinuity in p_0 there, but if it is large enough to make $p_0(x_0+, 0) < 0$, we shall call $x = x_0$ an artificial separation point. In either case y_0 becomes infinite at $x = x_0$ for $\psi > 0$ and the derivation of (2.3) is in error in neglecting, for example, $\epsilon\,(\partial^2 y/\partial x^2)$ compared with y for $x - x_0$ sufficiently small. Now Bernoulli's equation is

$$u^2 = \cos^2\phi(\zeta) + O(\epsilon\log p, u^2(\partial y/\partial x)^2),$$

and, for all the cases we shall consider, $\epsilon\log p$, y and $\partial y/\partial x$ will tend to zero either as $\epsilon \to 0$ or as $x \to x_0$.

Thus, in the neighbourhood of a separation point, we approximate (2.1*b*) by

$$\frac{\partial p}{\partial \psi} = \frac{\cos\phi(\zeta)}{R^j(x)}\left[\kappa - \frac{\partial^2 y}{\partial x^2}\right]. \tag{2.4}$$

Making similar approximations in the entropy equation (2.1*c*) and the boundary conditions (2.2) gives

$$\frac{\partial y}{\partial \psi} = \frac{\epsilon\sin^2\phi(\zeta)}{R^j(x)\cos\phi(\zeta)\left[\sin^2\phi(x) + \frac{1}{R^j(x)}\int_{\Psi(x)}^{\psi}\left(\kappa - \frac{\partial^2 y}{\partial x^2}\right)\cos\phi(\zeta)\,d\psi\right]},$$

or, if $y = \epsilon\bar{y}$,

$$\cos\phi(\zeta)\left[\kappa - \frac{\partial^2\bar{y}}{\partial x^2}\right] = \frac{d}{d\psi}\left(\frac{\sin^2\phi(\zeta)}{\cos\phi(\zeta)}\right)\frac{\partial\bar{y}}{\partial\psi} - \frac{\sin^2\phi(\zeta)}{\cos\phi(\zeta)}\frac{\partial^2\bar{y}}{\partial\psi^2}, \tag{2.5}$$

with boundary conditions

$$\bar{y} = 0 \quad \text{on} \quad \psi = 0, \tag{2.6a}$$

and

$$\frac{\partial\bar{y}}{\partial\psi} = \frac{1}{R^j(x)\cos\phi(x)} \quad \text{on} \quad \psi = \Psi(x). \tag{2.6b}$$

(2.6*b*) is just the condition for the streamlines to have the correct inclination to the shock in the Newtonian limit. The term in $\partial^2\bar{y}/\partial x^2$ in (2.5) is thus a perturbation of the trivial differential equation obtained by differentiating the unmodified first-order equation for y. It is only near the separation point that it has to be taken into account.

Since natural and artificial separation points require different mathematical techniques, we shall consider them separately.

3. Artificial separation

We shall just consider the simple case of a wedge of angle α smoothly joined at O to a cylinder of initial curvature κ (figure 2). The Newtonian theory of flow past slender bodies of this shape without separation has been studied by Cole

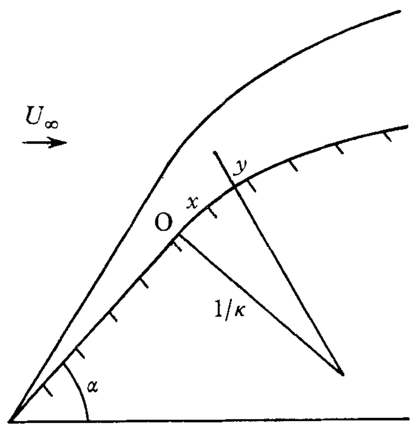

FIGURE 2. Curvature discontinuity on a wedge.

(1965), who found that for small x, ϵ can be eliminated by the scaling $X = x/\epsilon$. Making this substitution and letting $\epsilon \to 0$, (2.5) becomes, in the neighbourhood of O,

$$\left(\frac{\partial \bar{y}}{\partial \psi}\right)^2 \left(\kappa - \frac{\partial^2 \bar{y}}{\partial X^2}\right) = -\tan^2\alpha \frac{\partial^2 \bar{y}}{\partial \psi^2},$$

with $\bar{y} = 0$ on $\psi = 0$ and $\partial \bar{y}/\partial \psi = 1/\cos\alpha$ on $\psi = \psi_0$. Putting $Y = \bar{y} - \frac{1}{2}\kappa x^2$, this gives

$$\left(\frac{\partial Y}{\partial \psi}\right)^2 \frac{\partial^2 Y}{\partial X^2} = \tan^2\alpha \frac{\partial^2 Y}{\partial \psi^2} \tag{3.1}$$

with $\quad Y = -\frac{1}{2}\kappa X^2 \quad$ on $\quad \psi = 0, \qquad \partial Y/\partial \psi = 1/\cos\alpha \quad$ on $\quad \psi = \psi_0 \quad$ (3.2a,b)

and, from (2.3d) applied to the wedge,

$$Y = \psi/\cos\alpha, \quad \partial Y/\partial X = 0 \quad \text{on} \quad X = 0. \tag{3.2c,d}$$

In the (X, ψ)-plane, Cole found the solution downstream of the characteristic $X = \psi/\sin\alpha$, represented by OA in figure 3, in the parametric form

$$Y = -\tfrac{1}{2}\kappa X_0^2 + (\kappa\psi/\sin\alpha)\{X_0 - \tan\alpha/\kappa\}\exp(\kappa X_0/\tan\alpha), \tag{3.3a}$$

$$X = X_0 + (\psi/\sin\alpha)\exp(\kappa X_0/\tan\alpha). \tag{3.3b}$$

The reflected characteristic AB was found to intersect $\psi = 0$ provided $\lambda = \kappa\psi_0\cos\alpha - \sin^2\alpha < 0$, but, if $\lambda \geqslant 0$, the simple wave solution (3.3) persisted near the body, with the surface pressure being of order $\exp(-X)$. Since $(-\lambda)$ is the surface pressure just downstream of O as predicted by (2.3*b*), this result in itself provided some confirmation of the free-layer hypothesis. Finally, for the case $\lambda = 0$ the solution in the interaction region between AB and the shock wave was found to be

$$Y = \frac{\psi_0}{\cos\alpha}\log\frac{\psi}{\psi_0} - \frac{\kappa X^2}{2} + \tan\alpha X + \frac{\tan^2\alpha}{2\kappa}. \tag{3.4}$$

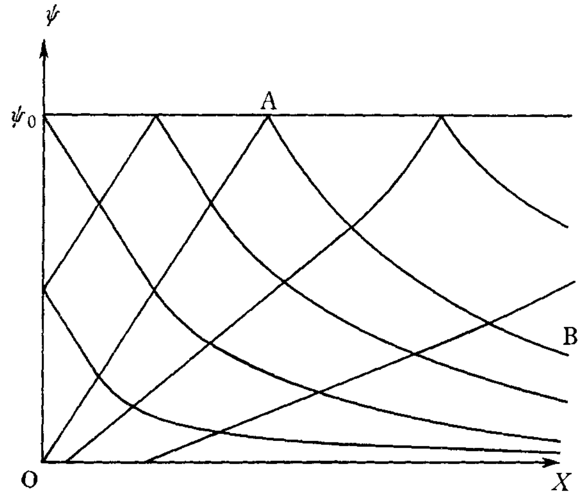

FIGURE 3. Characteristics in the (X, ψ)-plane.

We shall now consider the solution in the interaction region when $\lambda > 0$. Putting

$$\Phi(r,s) = X\frac{\partial Y}{\partial X} + \psi\frac{\partial Y}{\partial \psi} - Y,$$

where
$$r = -\tfrac{1}{4}\log\cos\alpha - \frac{1}{4\tan\alpha}\frac{\partial Y}{\partial X} - \tfrac{1}{4}\log\frac{\partial Y}{\partial \psi},$$

and
$$s = \tfrac{1}{4}\log\cos\alpha - \frac{1}{4\tan\alpha}\frac{\partial Y}{\partial X} + \tfrac{1}{4}\log\frac{\partial Y}{\partial \psi},$$

linearizes (3.1) to give

$$\frac{\partial^2\Phi}{\partial r\,\partial s} - \frac{\partial\Phi}{\partial r} + \frac{\partial\Phi}{\partial s} = 0.$$

The boundary condition on AB becomes, from (3.3),

$$\Phi(0,s) = -\frac{2\tan^2\alpha}{\kappa}s^2 \quad (s \geqslant 0),$$

while (3.2*b*) becomes

$$\frac{\partial\Phi}{\partial r} - \frac{\partial\Phi}{\partial s} = -\frac{4\psi_0}{\cos\alpha} \quad \text{on} \quad r = s.$$

Putting
$$w(r,s) = e^{r-s}J_0(2\surd\{(\eta-r)(\eta-s)\}),$$

$$\Lambda = \Phi\frac{\partial w}{\partial s} - w\frac{\partial\Phi}{\partial s} + 2\Phi w,$$

$$\Gamma = \Phi\frac{\partial w}{\partial r} - w\frac{\partial\Phi}{\partial r} - 2\Phi w,$$

and evaluating $\int \Lambda\, ds - \Gamma\, dr$ around the triangle $(0,0), (0,\eta), (\eta,\eta)$ gives

$$\Phi(\eta,\eta)+F(\eta) = 2\int_0^\eta \Phi(s,s)\, J_0(2(\eta-s))\, ds, \tag{3.5}$$

where

$$F(\eta) = \frac{4\tan^2\alpha}{\kappa}\int_0^\eta (2s-s^2)\, e^{-s} J_0(2\surd\{\eta(\eta-s)\})\, ds + \frac{4\psi_0}{\cos\alpha}\int_0^\eta J_0(2(\eta-s))\, ds.$$

This is an integral equation for the shock shape, for if $Y = Y_s$ on the shock, then

$$Y_s = (\psi_0/\cos\alpha) + X(dY_s/dX) - \Phi(\eta,\eta), \tag{3.6}$$

where

$$dY_s/dX = -4\tan\alpha\eta. \tag{3.7}$$

Substituting the exact solution (3.4) when $\lambda = 0$ into (3.5) gives the identity

$$-(\eta+2\eta^2) + \int_0^\eta (2s-s^2)\, e^{-s} J_0(2\surd\{\eta(\eta-s)\})\, ds + \int_0^\eta J_0(2(\eta-s))\, ds$$
$$= -2\int_0^\eta (s+2s^2)\, J_0(2(\eta-s))\, ds.$$

Hence, putting

$$\Phi(\eta,\eta) + \frac{4\tan^2\alpha}{\kappa}(\eta+2\eta^2) - 2\left(\frac{\psi_0}{\cos\alpha} - \frac{\tan^2\alpha}{\kappa}\right) = f(\eta),$$

(3.5) becomes

$$f(\eta)+\mu = 2\int_0^\eta f(s)\, J_0(2(\eta-s))\, ds, \tag{3.8}$$

where

$$\mu = 2\left(\frac{\psi_0}{\cos\alpha} - \frac{\tan^2\alpha}{\kappa}\right).$$

Differentiating (3.8) three times and using the identity

$$\int_0^\eta J_1(u)\, J_0(\eta-u)\frac{du}{u} = J_1(\eta)$$

gives

$$\eta\frac{d^3f}{d\eta^3} = -4\mu J_1(2\eta),$$

whence

$$\eta\frac{d^2f}{d\eta^2} - \frac{df}{d\eta} = 2\mu J_0(2\eta).$$

The shock shape is thus given parametrically by (3.7) and

$$\frac{d}{d\eta}\left(\frac{X}{\eta}\right) = -\frac{\tan\alpha}{\kappa\eta^2} - \frac{\mu}{2\tan\alpha}\frac{J_0(2\eta)}{\eta^2}.$$

This last equation may be integrated to give

$$X = \frac{\tan\alpha}{\kappa} + 2\left(\frac{\psi_0}{\sin\alpha} + \frac{\tan\alpha}{\kappa}\right)\eta + \frac{\mu}{2\tan\alpha}\left[J_0(2\eta) - 2\eta J_1(2\eta) + 4\eta\int_0^\eta J_0(2u)\, du\right], \tag{3.9}$$

where the constant of integration has been evaluated by expanding for small η and using (3.8). Finally, from (3.7) and (3.9),

$$Y_s = \frac{\psi_0}{\cos\alpha} - 4\tan\alpha\left(\frac{\psi_0}{\sin\alpha} + \frac{\tan\alpha}{\kappa}\right)\eta^2$$
$$-\mu\left[\eta J_0(2\eta) - 2\eta^2 J_1(2\eta) + 4(\eta^2-1)\int_0^\eta J_0(2u)\, du\right]. \tag{3.10}$$

Thus, as $X \to \infty$, $Y_s \to (\psi_0/\cos\alpha) - (\sin^2\alpha/2\psi_0\cos\alpha)X^2$ for all $\kappa > \sin^2\alpha/\psi_0\cos\alpha$ and so, as $\epsilon \to 0$ the shock shape near O tends to

$$y_s = \left(\frac{\kappa}{2} - \frac{\sin^2\alpha}{2\psi_0\cos\alpha}\right)x^2.$$

This is exactly the free-layer solution near O, as found by equating (2.3*b*) to zero and choosing the solution which touches the body at O.

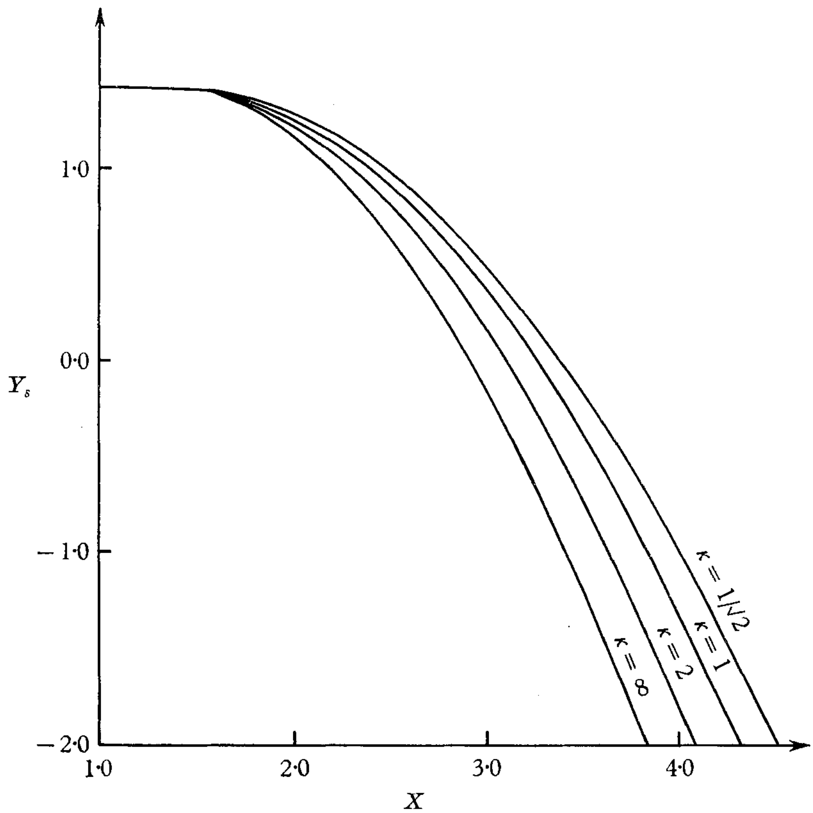

FIGURE 4. The shock waves in the (X, Y)-plane.

In figure 4, Y_s is plotted for different values of $\kappa > 1/\sqrt{2}$ with $\alpha = \frac{1}{4}\pi$ and $\psi_0 = 1$. The curves all tend to the same parabola, displaced different distances in the X-direction.

4. Natural separation

The uniformly valid solution at a natural separation point does not exhibit discontinuous derivatives and thus may be treated by normal singular-perturbation techniques. In contrast to the case considered above, the solution of (2.5) now differs from the unmodified solution (2.3*d*) everywhere, but only markedly so near the separation point.

Let us first consider the case of a sphere of radius a, taking the origin of the curvilinear co-ordinates at the upstream stagnation point. Putting $\theta = x/a$, $\xi = \zeta/a$, the separation point is at $\theta = \theta_0 = \frac{1}{3}\pi$ and (2.5) becomes

$$\sin^2\xi\left(\frac{\partial\bar{y}}{\partial\xi}\right)^2\left(1-\frac{\epsilon}{a}\frac{\partial^2\bar{y}}{\partial\theta^2}\right) = -a\cos\xi\left(3\sin\xi\frac{\partial\bar{y}}{\partial\xi}+\cos\xi\frac{\partial^2\bar{y}}{\partial\xi^2}\right), \tag{4.1}$$

with
$$\bar{y} = 0 \quad \text{on} \quad \xi = 0, \tag{4.2a}$$
and
$$\partial\bar{y}/\partial\xi = a\cot\theta \quad \text{on} \quad \theta = \xi. \tag{4.2b}$$

Equation (4.1) may be treated by Lighthill's (1949) technique, in which both the independent and dependent variables are expanded as power series in ϵ,

$$\bar{y}(\theta,\xi) = y_0(z,\xi)+\epsilon y_1(z,\xi)+\dots, \tag{4.3a}$$
$$\theta-\tfrac{1}{3}\pi = z+\epsilon x_1(z)+\dots. \tag{4.3b}$$

For our special case x_1 does not have to depend on ξ. Lighthill's hypothesis is that if $x_1, x_2 \dots$ are chosen so as to eliminate the worst singularities in $y_1, y_2 \dots$ respectively as $z \to 0$, then (4.3a) and (4.3b) will converge uniformly in a neighbourhood of $x = x_0$ which does not vanish as $\epsilon \to 0$. The solution for y_0 satisfying

$$y_0 = 0 \quad \text{on} \quad \psi = 0,$$
and
$$\partial y_0/\partial\xi = a\cot\xi \quad \text{on} \quad \xi = \tfrac{1}{3}\pi+z,$$
is
$$y_0 = 3a\int_0^{\xi}\frac{\cos^3\xi\,d\xi}{\sin^3\xi-\sin 3z}$$
$$= \frac{2\pi a}{3^{\frac{7}{6}}}(-z)^{-\frac{2}{3}}+O(\log|z|) \tag{4.4}$$

as $z\to 0$ for $\xi > 0$. The equation for y_1 is thus

$$\frac{\partial^2 y_1}{\partial\xi^2}+\left[3\tan\xi+\frac{6\cos\xi\sin^2\xi}{\sin^3\xi-\sin 3z}\right]\frac{\partial y_1}{\partial\xi} = \left(\frac{\partial y_0}{\partial\xi}\right)^2\frac{\partial^2 y_0}{\partial z^2}\frac{\tan^2\xi}{a^2}, \tag{4.5}$$

where we have neglected terms which are small compared with $\partial^2 y_0/\partial z^2$ as $z\to 0$. The boundary condition on the shock wave is

$$\frac{\partial}{\partial\xi}(\bar{y}-y_0) = -a\epsilon x_1(z)\operatorname{cosec}^2(\tfrac{1}{3}\pi+z)-\left[\frac{\partial y_0}{\partial\xi}-a\cot(\tfrac{1}{3}\pi+z)\right] \tag{4.6a}$$
on
$$\xi = \tfrac{1}{3}\pi+z+\epsilon x_1(z)+\dots, \tag{4.6b}$$

since, in (4.6b), x_1 will be large compared to y_0 as $z\to 0$. Thus we may take

$$\partial y_1/\partial\xi = ax_1(z)\,(4-\operatorname{cosec}^2(\tfrac{1}{3}\pi+z))$$
on
$$\xi = \tfrac{1}{3}\pi+z.$$

(4.5) may now be integrated to give

$$y_1 = 9\sin^2(\tfrac{1}{3}\pi+z)\cos(\tfrac{1}{3}\pi+z)\,(4-\operatorname{cosec}^2(\tfrac{1}{3}\pi+z))aI_1+I_2,$$

where
$$I_1 = \int_0^{\xi}\frac{\cos^3\xi\,d\xi}{(\sin^3\xi-\sin 3z)^2}$$
and
$$I_2 = \frac{1}{a^2}\int_0^{\xi}\frac{\cos^3\xi\,d\xi}{(\sin^3\xi-\sin 3z)^2}\left\{\int_{\frac{1}{3}\pi+z}^{\xi}\frac{(\sin^3\xi-\sin 3z)^2}{\cos^3\xi}\left(\frac{\partial y_0}{\partial\xi}\right)^2\frac{\partial^2 y_0}{\partial z^2}\tan^2\xi\,d\xi\right\}.$$

Now
$$I_1 = K(-z)^{-\frac{5}{3}} + O(z^{-1})$$
as $z \to 0$ for $\xi > 0$, where
$$K = 3^{-\frac{5}{3}} \int_0^\infty \frac{dy}{(1+y^3)^2}.$$
Also
$$\frac{\partial^2 y_0}{\partial z^2} = F(\sin\xi, z) + O((-z)^{-\frac{2}{3}}),$$
where
$$F(\sin\xi, z) = 54a\cos^2 3z \int_0^\xi \frac{\cos^3\xi\, d\xi}{(\sin^3\xi - \sin 3z)^3} = \frac{20\pi a}{3^{\frac{19}{6}}}(-z)^{-\frac{8}{3}} + O(z^{-2})$$
as $z \to 0$ for $\xi > 0$. Thus
$$\begin{aligned} I_2 &= 3\int_0^\xi \frac{\cos^3\xi\, d\xi}{(\sin^3\xi - \sin 3z)^2}\left[F(\sin\xi, z)\sin^3\xi - F(\sin(\tfrac{1}{3}\pi + z), z)\sin^3(\tfrac{1}{3}\pi + z)\right] \\ &\qquad + O((-z)^{-\frac{5}{3}}) \\ &= -\frac{5\pi a K}{2\,.\,3^{\frac{2}{3}}}(-z)^{-\frac{13}{3}} + O((-z)^{-\frac{11}{3}}) \end{aligned}$$
as $z \to 0$ for $\xi > 0$. In this limit, therefore,
$$y_1 = 9aKx_1(z)(-z)^{-\frac{5}{3}} - \frac{5\pi a K}{2\,.\,3^{\frac{2}{3}}}(-z)^{-\frac{13}{3}} + O\left((-z)^{-\frac{11}{3}}, \frac{x_1(z)}{z}\right),$$
and so we can eliminate the worst singularity in y_1 by choosing
$$x_1(z) = \frac{5\pi}{2\,.\,3^{\frac{8}{3}}}(-z)^{-\frac{8}{3}}.$$

Hence the first approximation to $\bar{y}$ which is uniformly valid in the range $0 \leqslant \theta \leqslant \frac{1}{3}\pi$ is
$$\bar{y} = 3a\int_0^\xi \frac{\cos^3\xi\, d\xi}{\sin^3\xi - \sin 3z},$$
where
$$\theta - \tfrac{1}{3}\pi = z + \frac{5\pi\epsilon}{2\,.\,3^{\frac{8}{3}}}(-z)^{-\frac{8}{3}}.$$

At $\theta = \frac{1}{3}\pi$ the first approximation to y is about $2\epsilon^{\frac{9}{11}}a$ for $\xi > 0$ independently of ξ. Physically this means that the streamlines bunch near the shock wave at $\theta = \frac{1}{3}\pi$. Only those streamlines for which $\psi = O(\epsilon^{\frac{3}{11}})$ lie between the shock and the body and in this region and on the body the pressure is of $O(\epsilon^{\frac{3}{11}})$. These orders of magnitude justify the assumptions made in deriving (4.1).

This bunching of the streamlines was first mentioned by Freeman (1960). His matching method gave the distance from the shock to the body at $\theta = \frac{1}{3}\pi$ as about $2.6\epsilon^{\frac{9}{11}}a$.

The above method is applicable to natural separation points on any two-dimensional or axisymmetric body which has a first-order shock-layer solution of type (2.3). In particular, for pointed bodies the orders of magnitude are not as awkward as for the sphere. Then the problem can be treated quite generally, assuming only that the body pressure is monotonically decreasing at the separation point. This approach gives the distance from the shock to the body there as $-\{\tan^2\alpha_0/3\kappa(x_0)\}\epsilon\log\epsilon$ to first order, where α_0 is the inclination of the body at the nose. This is in exact agreement with the results of Bausch (1962) who considered the problem by matching methods similar to those of Freeman.

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